A DELIGNE-MUMFORD STRATIFICATION OF ELLIPTIC COHOMOLOGY WITH LEVEL STRUCTURE

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Abstract. Level structure is an essential aspect of the classical theory of modular forms and elliptic curves, and it plays a similarly important role in elliptic cohomology and its applications to physics and stable homotopy theory. The classical theory doesn't carry over cleanly to the spectral case, however, because level structures and their associated isogenies can fail to be étale. This talk concerns a new approach I have developed for understanding elliptic cohomology with (possibly ramified) level structure. By studying the effect of isogenies on the dualizing line, I obtain a tractable invariant characterizing "how ramified" an isogeny is. This invariant can be used to filter the abstract moduli stack of isogenies between oriented elliptic curves, ultimately producing a stratification by Deligne-Mumford stacks.

Contents

1. Level Structure on Elliptic Curves

In the classical complex-analytic case, we can describe the moduli stack of elliptic curves as a global quotient orbifold: $\mathcal{M}_{ell} \cong \mathbb{H}//SL_2(\mathbb{Z}) = (\mathbb{H} \cup \overline{\mathbb{H}})//GL_2(\mathbb{Z})$. This moduli stack carries a line bundle ω of fiberwise differential 1-forms, and a *modular form of weight k* is a section of $\omega^{\otimes k}$.

This construction can be generalized. The moduli stack of elliptic curves over $\mathbb{Z}$ similarly carries a line bundle ω, allowing us to define modular forms over $\mathbb{Z}$; and if we instead take Lurie's moduli stack $\mathcal{M}_{ell}^{or}$ of *oriented* elliptic curves (that is, strict spectral elliptic curves with an isomorphism of their completion to the Quillen formal group), the analogue of ω is already included via the orientation and so we can simply define the $\mathbb{E}_\infty$-ring of *topological modular forms*, TMF, to be the global sections of the structure sheaf.

If you've seen modular forms before, though, you know that number theorists don't stop at plain ol' elliptic curves. Rather, they generalize to *elliptic curves with level structure*, with the sections of the analogous line bundle on the associated moduli stack called *modular forms with level structure*. For any congruence

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subgroup[1] $\Gamma \subset GL_2(\mathbb{Z})$, the moduli of elliptic curves with Γ-structure is defined as $(\mathbb{H} \cup \overline{\mathbb{H}})/\Gamma$, which is a $GL_2(\mathbb{Z})/\Gamma$-cover of $\mathcal{M}_{ell}$.

Roughly speaking, a level structure on an elliptic curve E (over a ring R) is the data of an isogeny $E \to E'$, with the level being the degree of the isogeny. For the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, b \equiv c \equiv 0 \bmod N \right\},$$

for example, a $\Gamma(N)$ level structure on an elliptic curve is a homomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \to E(R)$ whose image is equal to $E[N](R)$ as a Cartier divisor. We can think of this as a pair of isogenies with transverse kernels. For the closely-related subgroup $\Gamma_1(N)$, a level structure is a degree N isogeny together with a choice of generator for the kernel, and for the subgroup $\Gamma_0(N)$ it is a degree N isogeny whose kernel admits a generator after an fppf extension. (See [2] for more info.)

Level structures show up naturally in many contexts. For example, a $\Gamma_0(2)$-structure on a complex elliptic curve is the same as a spin structure, and consequently the universal elliptic genus is valued in the ring of modular forms with $\Gamma_0(2)$-structure. (Its cousin the Witten genus is valued in plain old modular forms, at least for string manifolds.) In number theory, level structures are essential to just about every aspect of the theory of modular forms, including the construction of Hecke operators, the philosophy of cusp forms, automorphic representations, the Langlands conjecture for GL_2, and the modularity theorem.

Level structures also play an important role in topology. For instance, the ring spectra $L_{K(2)}TMF_0(N)$ frequently appear in finite resolutions of the $K(2)$-local sphere, as they are closely related to (and sometimes equal to) fixed-point spectra of E_2 by the Morava stabilizer group. They're also useful for studying equivariant tmf: genuine S^1-equivariant tmf can be approximated by C_n-equivariant tmf, and we have a decomposition ([5])

$$TMF^{C_n}[1/n] \simeq \prod_{k|n} TMF_1(k)[1/n]$$

describing the fixed-point spectrum after inversion of n. Moreover, the inclusion of level structure allows us to more clearly demonstrate the relationship between tmf and K-theory: as shown by Hill-Lawson in [1] (extending work of Lawson-Naumann in [3]), there is a commutative diagram of tmf-algebras

$$\begin{array}{ccc} tmf_0(3)[1/3] & \longrightarrow & ko[1/3] \\ \downarrow & & \downarrow \\ tmf_1(3)[1/3] & \longrightarrow & ku[1/3], \end{array}$$

where the top map factors through the localization of the well-known map $tmf \to ko[[q]]$.

You'll notice, though, that everything here is localized. This isn't a coincidence. Level N structures will not, in general, be étale unless N is inverted. Over $\mathbb{Z}$ the moduli stacks $\mathcal{M}(\Gamma)$ aren't always Deligne-Mumford (though they are Artin), and while their maps to $\mathcal{M}_{ell}$ are finite and faithfully flat, they may be ramified. This is quite a problem, since we need things to be étale to lift them to spectral algebraic

[1] A congruence subgroup is a subgroup of $GL_2(\mathbb{Z})$ containing some $\Gamma(N)$, defined below.

geometry. (Put another way, preserving the orientation requires a map to be an isomorphism on formal completions, which is the case iff it's étale.) But if we invert the level N to fix this, we lose the p-local information for every prime p dividing N. If we try to incorporate *every* level structure, this will make us lose basically all the chromatic information we want to have!

My solution: define the moduli stack from scratch, and break it into easier-to-understand pieces.

2. The Abstract Moduli Stack and Linear Ramification

Definition 2.1. We write $Isog_R$ for the functor $\mathrm{CAlg}_R \to \mathcal{S}$ sending an algebra A to the space of (*not necessarily orientation-preserving*) isogenies of oriented elliptic curves over A. We call this the *moduli stack of oriented elliptic curves with level structure.*

It's not too hard to show that this is a sheaf in the étale topology, but for the sake of time I'll ask you to take my word for it. It isn't known to be any kind of *geometric* stack, though, and it probably isn't one. (It's probably not Deligne-Mumford, at least.) Working with it therefore requires us to break it up into pieces that are more manageable. We need a couple of tools to do this, the first one being a way to measure the ramification explicitly.

Definition 2.2. Let $f : E \to E'$ be an isogeny of oriented elliptic curves. We get an induced map $\hat{f}$ on their completions, and since they are oriented, we can identify it with an endomorphism of $\widehat{\mathbb{G}}^Q_R$. Linearizing this gives us a module endomorphism of the so-called *dualizing line* (aka $\mathbb{E}_1$ cotangent space) $\omega_{\widehat{\mathbb{G}}^Q_R}$, which is canonically isomorphic to $\Sigma^{-2}R$. Suspending twice, we obtain a module endomorphism of R itself, and therefore an element $r \in \pi_0 R$. This gives us a map $\mathrm{ram}_1 : Isog_R \to \mathbb{A}^1_R$ in $\mathcal{S}\mathrm{hv}^{\acute{e}t}(\mathrm{CAlg})$ which we call *linear ramification.*

Let's write $Isog^{(r)}$ for the substack of isogenies with linear ramification equal to a unit times r, and $Isog^r_R$ for the substack of isogenies with ramification exactly r. (More precisely, this sends an R-algebra A to the subspace of isogenies whose linear ramification is (a unit times) the image of r, r_A. This is also an étale sheaf.) I claim that these are Deligne-Mumford stacks. This claim rests on two crucial results.

Theorem 2.3. *$Isog^{(1)}$ is a Deligne-Mumford stack.*

Theorem 2.4. *There is an orthogonal factorization system $(\mathcal{L}, \mathcal{R})$ on the category of isogenies of elliptic curves such that*

- *$\mathcal{L}$ is the class of morphisms inducing an isomorphism on étale fundamental groups, and*
- *$\mathcal{R}$ is the class of étale isogenies.*

To summarize, the second theorem essentially follows in the same way that one proves the corresponding result for Lie groups: take the connected-étale sequence of the kernel, and use this sequence and the 9-lemma to construct the factorization.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K^0 & = & K^0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K^{\grave{e}t} & \longrightarrow & E/K^0 & \longrightarrow & E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Another application of the 9-lemma shows that this factorization is unique.

The first theorem is proven using a theorem of Lurie's that I call "spectral deformation of representability". (This result is Theorem 18.1.0.2 in [4], and is one form of Artin-Lurie representability.)

Theorem 2.5 (Spectral Deformation of Representability)**.** *Let $X : \mathrm{CAlg}_S^{\mathrm{cn}} \to \mathcal{S}$ be a functor. Then X is a spectral Deligne-Mumford stack if and only if the following four conditions are satisfied:*

i) $X|_{\mathrm{CAlg}_S^{\heartsuit}}$ *agrees with* $Y|_{\mathrm{CAlg}_S^{\heartsuit}}$ *for some spectral Deligne-Mumford stack* Y*.*
ii) X *admits a cotangent complex.*
iii) X *is nilcomplete.*
iv) X *is infinitesimally cohesive.*

Proof that $Isog^{(1)}$ is a DM-stack. The proof of condition (ii) is rather technical, but the main point is that it can be reduced to the classical case. Condition (iii) is nilcompleteness, meaning that $X(A) = \lim_n X(\tau_{\leq n} A)$; this follows from the fact that étale subgroups are invariant under truncation of the base ring. Condition (iv), infinitesimal cohesion, essentially checks that X is actually a formal moduli problem. In our case, comes down to checking that a certain kind of pullback of $\mathbb{E}_\infty$-algebras induces a pullback on the corresponding spaces of étale subgroups, which is proven by a Nakayama's lemma argument relying on the fact that base change preserves exact sequences of strict abelian varieties.

The most interesting condition is probably (i), since it can be proven by showing that the restriction of X to classical rings is a classical DM-stack. Since the moduli of subgroups of rank N in an elliptic curve E is representable by an affine scheme ([2]), and we can take the coproduct over all N, we're reduced to showing that we can effectively "cut out" the étale subgroups from the moduli of all finite subgroups. Fortunately, this is the case: if we write G° for the connected component of the universal finite subgroup, the complement of the support of G° represents the classical moduli problem, showing that it is representable by a quasi-affine scheme. □

Putting these two results together, we can show that $Isog^{(r)}$ is DM as follows. For any elliptic curve E with an isogeny of ramification (r) out of it, we can uniquely factor this as $E \to E_r \to E'$, where the second map is étale and the first is the initial isogeny out of E with ramification equivalent to r. Consequently, $Isog(E/r)$ is equivalent to $Isog(E_r/(1))$, which is a closed substack of $Isog^{(1)}$ and thus a

DM-stack. The stacks $Isog(E/(r))$ (ranging over all oriented elliptic curves over R-algebras) form an étale cover of $Isog^{(r)}$, so it's covered by DM-stacks and is thus DM. Now just note that $Isog^r$ is a closed substack of $Isog^{(r)}$.

Finally, a bit of algebraic geometry shows that the space of points of $Isog$ is, as one would expect, the union of the spaces of points of $Isog^r$ as r varies. The union isn't disjoint because of the way points work (e.g. $\mathbb{A}^1$ isn't the disjoint union of the substacks corresponding to the elements of the base ring), but I'm currently working on giving a topos-theoretic description of the stratification. My most recent result is as follows. Here $Isog^{\leq r}$ is the moduli of isogenies whose linear ramification divides r.

Theorem 2.6. *The pro-étale topos of $Isog$ admits a stratification (in the sense of Barwick-Glasman-Haine)*

$$Isog^{(1)} \hookrightarrow \cdots \hookrightarrow Isog^{\leq r} \hookrightarrow \cdots \hookrightarrow Isog^{\leq 0} = Isog$$

over $\mathrm{Prin}(R)$ *whose strata are of the form $Isog^{(r)}$.*

References

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